COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 15 (3) (1995) 357–364

NON-CONTRACTIBLE EDGES IN A 3-CONNECTED GRAPH

YOSHIMI EGAWA, KATSUHIRO OTA, AKIRA SAITO[1] and XINGXING YU[2]

Received March 20, 1993
Revised October 5, 1993

An edge e in a 3-connected graph G is contractible if the contraction of e in G results in a 3-connected graph; otherwise e is non-contractible. In this paper, we prove that the number of non-contractible edges in a 3-connected graph of order $p \geq 5$ is at most

$$3p - \left\lfloor \frac{3}{2}(\sqrt{24p+25} - 5) \right\rfloor ,$$

and show that this upper bound is the best possible for infinitely many values of p.

1. Introduction

We follow the notation of Bondy and Murty [3]. Graphs discussed here are simple, i.e., with no loops or multiple edges. For a graph G, $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. If $S \subset V(G)$ or $S \subset E(G)$, then we use $G[S]$ to denote the subgraph of G induced by S. For $v \in V(G)$, let $N(v)$ denote the neighborhood of v in G. If A and B are two disjoint subsets of $V(G)$, then $[A,B]$ denotes the set of edges with one end in A and the other in B. For later convenience, $V_3(G)$ denotes the set of vertices of degree three in G; and for an edge e in G, $V(e)$ denotes the set of two endvertices of e.

The *contraction* of an edge $e = uv$ in a graph G is to identify the two vertices u and v and to delete all resulting loops and multiple edges; the resulting new vertex is denoted by e^* and the resulting new graph is denoted by Ge (sometimes also denoted by Guv). Hence, $V(Ge) = (V(G) - V(e)) \cup \{e^*\}$. Let σ_e be the map from $V(G)$ to $V(Ge)$ which takes both u and v to e^* and fixes all other vertices. Obviously, σ_e is a graph homomorphism, i.e., σ_e preserves adjacency. Thus, when we speak of the image of an edge of G under the contraction of e, we mean the image of the edge under σ_e.

An edge e in a 3-connected graph G is *contractible* if Ge is also 3-connected; otherwise, we say that e is *non-contractible*. Tutte [9] invented the concept of a contractible edge, and from his constructive characterization of 3-connected graphs, it is clear that every non-K_4 3-connected graph has a contractible edge. Many

---

Mathemacics Subject Classification (1991): 05 C 40

[1] Partially supported by Nihon University Research Grant B90-026

[2] Partially supported by NSF under grant No. DMS–9105173

0209–9683/95/$6.00

applications of contractible edges have been given, including Plummer and Toft [7], Thomassen [8], and Yu [11]. In most situations, we need to have more than one contractible edge. In [2], Ando, Enomoto and Saito proved that every non-K_4 3-connected graph G has at least $|V(G)|/2$ contractible edges, and they also characterized all extremal graphs. (For a good reference on contractible edges, see [4].) Non-contractible edges (like contractible edges) have been used in several papers, for example, [1] and [10]. The purpose of this paper is to give a sharp upper bound to the number of non-contractible edges in a 3-connected graph.

It is not hard to see that in a non-K_4 3-connected graph G, an edge e is non-contractible if and only if $V(e)$ is contained in some 3-cut S of G; in this case, we say that S is a 3-cut associated with e. Let G be a 3-connected graph. Then $E_c(G)$ and $E_n(G)$ denote the set of contractible edges and the set of non-contractible edges in G, respectively. For a vertex u in G, let $E(u)$ denote the set of edges in G incident to u, let $E_c(u)=E(u)\cap E_c(G)$, and let $E_n(u)=E(u)\cap E_n(G)$. Also, let $e_n(G)=|E_n(G)|$. Now we can state our main result.

Theorem. *Let G be a non-K_4 3-connected graph on p vertices. Then*

$$e_n(G) \leq 3p - \left\lfloor \frac{3}{2}(\sqrt{24p+25}-5) \right\rfloor .$$

Our result is the best possible in the sense that there are arbitrarily large graphs attaining this bound. In fact, some extremal graphs arise naturally from the proof. We also point out that the technique we will use may have its potential in studying the distributions of contractible edges (for example, see [1] and [10]).

2. Preliminary results

To prove the main result, we need a few lemmas. The first one is due to Ando, Enomoto and Saito [2], and implies the existence of a contractible edge at any vertex of degree three in a non-K_4 3-connected graph.

Lemma 2.1. *Let G be a non-K_4 3-connected graph, and let u be a vertex of G with $N(u)=\{s,x,y\}$ such that ux, $uy \in E_n(G)$. Then, $d(x)=d(y)=3$, $G[\{u,x,y\}]=K_3$, and $E_c(u)=\{su\}$.* ∎

Lemma 2.2. *Let G be a non-K_4 3-connected graph, and let $u\in V_3(G)$ with $N(u)=\{x,y,s\}$ and $xy\in E(G)$. Then $su\in E_c(G)$.*

Proof. It is easy to check that $\{s,u\}$ cannot be contained in any 3-cut of G. ∎

The essential part of the following lemma was first proved by the fourth author in [10], but for the sake of completeness, we also give the proof here.

Lemma 2.3. *Let G be a non-K_4 3-connected graph, and let $e \in E_c(G)$ and $f \in E_n(G)$. Then, $\sigma_e(f)\in E_c(Ge)$ if and only if there is a vertex $u\in V(e)$ with $d(u)=3$ such that $N(u)$ is the only 3-cut of G associated with f and such that $G-N(u)$ has exactly two components.*

Proof. For $\sigma_e(f)\in E_c(Ge)$, the contraction of e in G must destroy all 3-cuts of G associated with f. Let S be a 3-cut of G associated with f. Since $e\in E_c(G)$, $e\in$

$E(C)\cup[V(C),S]$ for some component C of $G-S$. Clearly, $\sigma_e(f)\in E_c(Ge)$ if and only if $|V(C)|=1$, S is the only 3-cut of G associated with f, and $G-S$ has exactly two components. Lemma 2.3 now follows. ∎

Following [1], we call an edge f a *turncoat* of G via a contractible edge e if $f\in E_n(G)$ and $\sigma_e(f)\in E_c(Ge)$. We use Lemma 2.3 here to prove the following result which will be used in the proof of our main theorem.

Lemma 2.4. *Let G be a non-K_4 3-connected graph, and let $e=uv\in E_c(G)$ such that e is contained in at most two triangles. Then $e_n(G)\le e_n(Ge)+3$. Moreover, the equality holds if and only if there is a vertex of $V(e)$, say u, such that $G[N(u)]=K_3$, $N(u)$ is the only 3-cut of G associated with some edge of $G[N(u)]$, and $G-N(u)$ has exactly two components.*

Proof. Suppose first that $d(u)\ge 4$ and $d(v)\ge 4$. Then by Lemma 2.3 G has no turncoat via e. Since e is contained in at most two triangles, we have $e_n(G)\le e_n(Ge)+2$. Hence we may assume that $d(u)=3$. We have three cases based on the number of triangles containing e.

Case 1. There are two triangles containing e. Then $d(v)\ge 4$ by 3-connectedness of G. Let x and y be the other vertices of the two triangles containing e. It is not hard to see, by Lemma 2.2, that $E_c(u)=E(u)$. Thus vx, vy and xy are the only possible turncoats of G via e. Hence $e_n(G)\le e_n(Ge)+3$, where the equality holds if and only if vx, vy and xy are turncoats via e. Lemma 2.4 now follows from Lemma 2.3.

Case 2. There is exactly one triangle containing e. Let the other vertex of the triangle be x, and let y be the third neighbor of u. If $d(v)=3$, then let z be the third neighbor of v. By Lemma 2.3, all possible turncoats of G via e are ux, vx, xy and zx. But in Ge, $\{x,y,z\}$ is a 3-cut. Hence G has at most two turncoats via e; namely, ux and vx. Thus $e_n(G)\le e_n(Ge)+2$.

So let $d(v)\ge 4$. By Lemma 2.3 ux cannot be a turncoat of G via e. Since $vy\notin E(G)$, G has at most two turncoats via e; namely, vx and xy. If vx is a turncoat via e, then $ux\in E_c(G)$, and so, $e_n(G)\le e_n(Ge)+2$. Hence, suppose that vx is not a turncoat via e. Then $e_n(G)\le e_n(Ge)+2$.

Case 3. No triangle contains e. Then no two edges of G have the same image after contracting e. By Lemma 2.3, G has at most two turncoats via e. Hence $e_n(G)\le e_n(Ge)+2$. ∎

In order to state our next lemma, we need some more notation. Suppose that $f=xy\in E_n(G)$ in a 3-connected graph G, and that $S=\{x,y,s\}$ is a 3-cut of G associated with f. Let $C\ne G-S$ be a non-empty union of components of $G-S$. We define a new graph C' from C as follows:

$$V(C')=V(C)\cup S\cup\{i\},$$
$$E(C')=E(C)\cup[V(C),S]\cup\{ix,iy,is,xy,ys,xs\}.$$

Obviously, C' is 3-connected. The following result was proved by the fourth author in [10] for the case when C is a single component. However, the proof for the general case here can be easily derived from Lemma 2.5 of [10].

Lemma 2.5. *Let $|V(C)| \geq 2$. For every $e \in E(C) \cup [V(C), S]$, $e \in E_c(C')$ if and only if $e \in E_c(G)$ with only one exceptional case: $G - S$ has exactly two components, $V(e) \subset V(C) \cap N(s)$ and $d(s) = 3$, and $N(s)$ is the only 3-cut of G associated with e.* ∎

It is clear by Lemma 2.2 that all edges at i are contractible in C'. Also note that when the exceptional case of Lemma 2.5 occurs to C, we have $|E(G[S])| = 1$. These two observations will be used later to count the non-contractible edges.

3. Small graphs

For simplicity, let

$$g(x) = 3x - \left\lfloor \frac{3}{2}(\sqrt{24x + 25} - 5) \right\rfloor .$$

By an easy argument, we can show that $g(x) \geq 2x - 7$ for $x \geq 14$. Throughout the rest of the paper, we use G to denote a 3-connected graph with $p \geq 5$ vertices. In this section, we show that $e_n(G) \leq g(p)$ for graphs with at most 20 vertices. The following table shows some values of $g(p)$ (for $p \leq 13$).

p	5	6	7	8	9	10	11	12	13
$g(p)$	5	6	8	10	12	14	15	17	19

We first observe that there are only three non-isomorphic 3-connected graphs on five vertices; namely, the wheel W_4, the complete graph K_5, and $K_5 - e$. Hence, it is easy to see that $e_n(G) \leq 4$ when $p = 5$.

Lemma 3.1. *If $p = 6$, then $e_n(G) \leq 6$. Moreover, the equality holds if and only if $G = K_3 \times K_2$.*

Proof. We may assume that $e_n(G) \neq 0$. Let $f = xy \in E_n(G)$, and let $S = \{x, y, s\}$ be a 3-cut of G. Then $G - S$ has two or three components. If $G - S$ has three components, then each component is a single vertex, and it is clear that $e_n(G) \leq 3$. So $G - S$ has exactly two components: a single vertex component $\{u\}$, and a component $C = \{v, w\}$. Since G is 3-connected, without loss of generality, we may assume that $xv, yw, sw \in E(G)$.

If $xw, yv, sv \in E(G)$, then there are four cases (whether or not $xs, ys \in E(G)$). In each case, it is easy to check that $e_n(G) \leq 3$.

Suppose that exactly one of xw, yv or sv does not belong to $E(G)$. First, let $sv \notin E(G)$. By 3-connectedness and symmetry, let $ys \in E(G)$. Note that xs may or may not be in $E(G)$, and in each case, $e_n(G) = 5$. Let $sv \in E(G)$, and by symmetry, let $xw \notin E(G)$. We have four cases (whether or not xs, $ys \in E(G)$), and in each case we have $e_n(G) \leq 5$.

Hence, we may assume that at least two of xw, yv or sv do not belong to $E(G)$. By 3-connectedness exactly two of xw, yv or sv are not in $E(G)$. Since the degree of v is at least three, $xw \notin E(G)$. First, let $yv \in E(G)$. In this case, we have three cases (whether or not xs, $ys \in E(G)$), and it is easy to check that $e_n(G) = 5$.

Hence, let $sv \in E(G)$. Again, there are four possible cases (whether or not xs, $ys \in E(G)$). In the case that $xs, ys \notin E(G)$, we have $G = K_3 \times K_2$ and $e_n(G) = 6$, and for all other cases, $e_n(G) = 5$. Hence Lemma 3.1 follows. ∎

Lemma 3.2. *If* $6 \le p \le 20$, *then* $e_n(G) \le 2p-7$ *except* $G = K_3 \times K_2$.

Proof. By Lemma 3.1, we may assume that $p \ge 7$. Let G be a counter-example to Lemma 3.2 so that $|V(G)|+|E(G)|$ is minimum. We first point out that for each $e \in E_c(G)$, e is contained in at most two triangles; for otherwise, $G-e$ is 3-connected, and so, $e_n(G) \le e_n(G-e) \le 2p-7$, a contradiction. (Hence, Lemma 2.4 can be applied.)

Claim 1. *For each* $e \in E_c(G)$, $e_n(G) = e_n(Ge)+3$.

For otherwise, $e_n(G) \le e_n(Ge)+2$. If $Ge \ne K_3 \times K_2$, then $e_n(G) \le 2p-7$, a contradiction. So $Ge = K_3 \times K_2$. Thus, the two vertices of $V(e)$ must have degree three in G and must share a common neighbor in G. It is not difficult to see that there are only two such (non-isomorphic) graphs, and in each case $e_n(G) \le 6$, a contradiction. Hence, Claim 1 follows.

Let $m = |V(G) - V_3(G)|$. Then, by Claim 1 and Lemma 2.4, $\frac{m(m-1)}{2} \ge e_n(G) \ge 2p-6$, and so,

$$m \ge \frac{1+\sqrt{16p-47}}{2}.$$

Claim 2. *For each* $f \in E_n(G)$ *and for any 3-cut* S *of* G *associated with* f, $G-S$ *has exactly two components one of which is a single vertex.*

For otherwise, we can divide the components into two disjoint parts, say C and D, each with at least two vertices. By Lemma 3.1 and the choice of G, $e_n(C') \le 2(|C|+4)-7$ and $e_n(D') \le 2(|D|+4)-7$. Now by Lemma 2.5 and by the observation following Lemma 2.5, $e_n(G) \le e_n(C') + e_n(D') - 3 \le 2p-7$, a contradiction.

By Claims 1 and 2, we have $3(p-m) = e_n(G) \ge 2p-6$, and so $m \le \frac{p+6}{3}$. Hence we have

$$\frac{1+\sqrt{16p-47}}{2} \le \frac{p+6}{3},$$

and so, $p \ge 21$ or $p \le 6$, a contradiction which completes the proof. ∎

To see that Lemma 3.2 is the best possible, let W_{p-1} be the wheel on p vertices ($6 \le p \le 20$). Construct a graph G from W_{p-1} by joining a non-center vertex v to all other vertices except one which is distance 2 away from v on the rim. Then, $e_n(G) = 2p-7$. Since $g(x) \ge 2x-7$ for $x \ge 14$, by checking values of $g(p)$ for $p \le 13$, we have the following consequence.

Corollary 3.3. *If* $5 \le p \le 20$, *then* $e_n(G) \le g(p)$. ∎

4. Proof of the main result

We state without proof the following elementary lemma.

Lemma 4.1. *Let q be an integer, and let a, b and c be three real numbers. If $a-b\leq q$, then $\lfloor a\rfloor - \lfloor b\rfloor \leq q$; if $a+b-c\geq q$, then $\lfloor a\rfloor + \lfloor b\rfloor - \lfloor c\rfloor \geq q-1$.*

Proof of Theorem. Suppose that this result is not true. Let G be a counter-example with $|V(G)|+|E(G)|$ minimum. By Corollary 3.3, we may assume that $p\geq 21$. As shown in the proof of Lemma 3.2, every $e\in E_c(G)$ is contained in at most two triangles.

Claim 1. *For any $e\in E_c(G)$, $e_n(G)=e_n(Ge)+3$.*

Suppose that $e_n(G)\leq e_n(Ge)+2$. Since $p\geq 21$,

$$\frac{3}{2}(\sqrt{24p+25}-5)-\frac{3}{2}(\sqrt{24(p-1)+25}-5)=\frac{3}{2}\frac{24}{\sqrt{24p+25}+\sqrt{24(p-1)+25}}<1.$$

Hence by Lemma 4.1

$$d=\left\lfloor\frac{3}{2}(\sqrt{24p+25}-5)\right\rfloor-\left\lfloor\frac{3}{2}(\sqrt{24(p-1)+25}-5)\right\rfloor\leq 1.$$

Therefore

$$e_n(G)\leq g(p-1)+2=g(p)+d-1\leq g(p).$$

This is a contradiction, and so, Claim 1 is true.

Claim 2. *For every $f=xy\in E_n(G)$ and every 3-cut $S=\{x,y,s\}$ associated with f, $G-S$ has exactly two components one of which is a single vertex.*

Suppose that Claim 2 is not true. Since G has $p\geq 21$ vertices, we may divide the components of $G-S$ into two disjoint parts, say C and D, such that $2\leq |C|=x\leq |D|=p-x-3$. Then, $2\leq x\leq \frac{p-3}{2}$. By the choice of G, $e_n(C')\leq g(x+4)$ and $e_n(D')\leq g(p-x+1)$. By Lemma 2.5 and the observation following Lemma 2.5, $e_n(G)\leq e_n(C')+e_n(D')-3$.

Suppose that $2\leq x\leq 16$. Then, $6\leq |C'|\leq 20$, and so, by Lemma 3.2, $e_n(C')\leq 2|C'|-7=2x+1$. Hence, $e_n(G)\leq 2x+g(p-x+1)-2$. Since $p\geq 21$ and $x\leq\frac{p-3}{2}$, $p-x+1\geq 13$, and so,

$$\frac{3}{2}(\sqrt{24p+25}-5)-\frac{3}{2}(\sqrt{24(p-x+1)+25}-5)$$

$$=\frac{3}{2}\frac{24(x-1)}{\sqrt{24p+25}+\sqrt{24(p-x+1)+25}}<x-1.$$

Hence by Lemma 4.1

$$d'=\left\lfloor\frac{3}{2}(\sqrt{24p+25}-5)\right\rfloor-\left\lfloor\frac{3}{2}(\sqrt{24(p-x+1)+25}-5)\right\rfloor\leq x-1.$$

Therefore we have

$$e_n(G) \leq 2x + g(p-x+1) - 2 = g(p) + d' - (x-1) \leq g(p).$$

Again, this is a contradiction. Thus, we may assume that $x \geq 17$, and so, $p \geq 37$ and $p-x+1 \geq 21$.

Note that the function $f(x) = \sqrt{24(x+4)+25} + \sqrt{24(p-x+1)+25} - \sqrt{24p+25}$ is an increasing function on the interval $[17, \frac{p-3}{2}]$. Thus,

$$f(x) \geq f(17) = 23 - \frac{24 \times 16}{\sqrt{24(p-17+1)+25} + \sqrt{24p+25}} > 15.$$

Hence, by Lemma 4.1,

$$\begin{aligned} d'' &= \left\lfloor \frac{3}{2}(\sqrt{24(x+4)+25} - 5) \right\rfloor + \left\lfloor \frac{3}{2}(\sqrt{24(p-x+1)+25} - 5) \right\rfloor \\ &\quad - \left\lfloor \frac{3}{2}(\sqrt{24p+25} - 5) \right\rfloor \\ &\geq \frac{3}{2}(\sqrt{24(x+4)+25} - 5) + \frac{3}{2}(\sqrt{24(p-x+1)+25} - 5) \\ &\quad - \frac{3}{2}(\sqrt{24p+25} - 5) - 1 \\ &\geq \frac{3}{2}(f(x) - 5) - 1 \geq \frac{3}{2}(f(17) - 5) - 1 > 14. \end{aligned}$$

Therefore, we have

$$e_n(G) \leq e_n(C') + e_n(D') - 3 \leq g(x+4) + g(p-x+1) - 3 = g(p) - d'' + 12 < g(p).$$

But this contradicts the choice of G. Hence, Claim 2 follows.

Following Lemma 2.4 and Claim 1, we have the following observation: for each $e \in E_c(G)$, there is a vertex $u \in V_3(G)$ such that $G[N(u)] = K_3$. Thus, by Claim 2, $E_n(G)$ is the disjoint union of $E(G[N(u)])$ for all $u \in V_3(G)$. Let $|V_3(G)| = k$ and let $m = p-k$. Then $e_n(G) = 3k = 3p - 3m$. Now $G - V_3(G)$ is a graph with m vertices and $3p - 3m$ edges. Hence, we have $\frac{m(m-1)}{2} \geq 3p - 3m$. By solving this inequality for m, we have $m \geq \frac{\sqrt{24p+25}-5}{2}$. Thus, $e_n(G) \leq g(p)$, a contradiction which ends the proof. ∎

From the above proof, it is easy to see that for some extremal graphs G, $E_n(G)$ may be the edge-disjoint union of triangles which are the neighborhoods of vertices in $V_3(G)$. Hence, we can simply start with a complete graph K_m and decompose $E(K_m)$ into as many edge-disjoint triangles as possible, and for each such triangle, we add a new vertex and join it to the three vertices of the triangle. For example, in [6] it was shown that $E(K_{6k+1})$ can be decomposed into edge-disjoint triangles. The graphs obtained by adding a degree 3 vertex to each such triangle achieve the bound in the theorem.

Actually, we can give a slightly better bound

$$\left\lfloor 3(p - \frac{\sqrt{24p+25}-5}{2}) \right\rfloor$$

to $e_n(G)$ by further developing the arguments which we have used in this paper. However, its proof is lengthy and hence we omit it.

Acknowledgment. The fourth author thanks Richard Duke for some helpful comments.

References

[1] R. E. L. ALDRED, R. L. HEMMINGER, and X. YU: The 3-connected graphs with a maximum matching containing precisely one contractible edge, *J. Graph Theory*, in press.

[2] K. ANDO, H. ENOMOTO, and A. SAITO: Contractible edges in 3-connected graphs, *J. Comb. Theory Ser. B* **42** (1987), 87–93.

[3] J. A. BONDY, and U. S. R. MURTY: *Graph Theory with Applications*, American Elsevier (1976).

[4] N. DEAN: Contractible edges and conjectures about path and cycle numbers, *Ph.D Thesis*, Vanderbilt University (1987).

[5] R. L. HEMMINGER, and X. YU: 3-Connected graphs with contractible edge covers of size k, *Discrete Math.* **101** (1992), 115–133.

[6] T. B. KIRKMAN: On a problem in combinations, *Cambridge and Dublin Math. J.* **2** (1847), 191–204.

[7] M. D. PLUMMER, and B. TOFT: Cyclic colorations of 3-polytopes, *J. Graph Theory* **11** (1987), 507–515.

[8] C. THOMASSEN: Planarity and duality of finite and infinite graphs, *J. Comb. Theory Ser. B* **29** (1980), 244–271.

[9] W. T. TUTTE: A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441–455.

[10] X. YU: 3-Connected graphs with non-cut contractible edge covers of size k, *J. Graph Theory*, in press.

[11] X. YU: Non-separating cycles and discrete Jordan curves, *J. Comb. Theory Ser. B* **54** (1992), 142–154.

Yoshimi Egawa

Department of Applied Mathematics
Science University of Tokyo
Shinjuku-ku, Tokyo 162, JAPAN

Katsuhiro Ota

Department of Mathematics
Keio University
Hiyoshi 3-14-1, Kohoku-ku
Yokohama-shi, Kanagawa 223, JAPAN
ohta@comb.math.keio.ac.jp

Akira Saito

Department of Mathematics
Nihon University
Sakurajosui 3-25-40
Setagaya-ku, Tokyo 156, JAPAN
asaito@math.chs.nihon-u.ac.jp

Xingxing Yu

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332, USA
yu@math.gatech.edu